SOME RESTRAINTS ON THE ENERGY AND SPECTRUM OF TURBULENT FLUID MOTION

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A thermodynamically nonequilibrium fluid tends to return to its stable equilibrium state owing to the dissipation and redistribution of part of its energy as a result of individual particle collisions. However, if the fluid deviates strongly from the state of thermodynamic equilibrium, it is more advantageous for it to go over into a state of random motion, which permits the more rapid liquidation of the nonequilibrium condition in the active development of turbulent transfer processes exceeding in magnitude the classical processes of molecular transfer. Finding the spectrum and energy level of the resulting turbulent motion is a complex and, in some cases, mathematically impossible task. Therefore it is useful to have at least some restraints on their magnitudes, in order to be able to estimate the role of turbulent transfer processes in the pattern of evolution of the unstable state of the fluid. As these conditions we shall take the conditions of stability of the steady turbulent state of the fluid.

In accordance with ideas of L. D. Landau [1], the motion of a fluid in the developed turbulence regime can be thought of as a certain quasi-periodic motion and the physical quantities may be described in the form of a sum of periodic functions with different periods.[1]

$$\psi^\alpha(x, y, z, t) = \sum_{p_1,\ldots,p_n=-\infty}^{+\infty} A^\alpha_{p_1\ldots p_n}(x, y, z) \times$$
$$\times \exp\left[-\sum_{l=1}^{n} i p_l(\omega_l t + \varphi_l)\right]. \qquad (1)$$

Here ψ^α (x, y, z, t) is the vector of state, whose components are parameters characterizing this state (hydrodynamic velocity $\psi^1 \equiv v_x$, pressure $\psi^2 = p$, and so on), ω_l is the frequency, and φ_l the phase of the individual periodic motions.

For steady-state turbulence, the parameters characterizing the flow depend on time; therefore for small deviations from this state the coefficients in the equations of hydrodynamics will depend on time. These equations may be written in the form

$$\frac{\partial}{\partial t}\delta\psi^\alpha(\mathbf{r}, t) = \sum_\beta H^{\alpha\beta}\left(\frac{\partial}{\partial \mathbf{r}}, \psi\right)\delta\psi^\beta(\mathbf{r}, t). \qquad (2)$$

Here $H^{\alpha\beta}$ is a differential operator depending on ψ^α. We shall assume that the turbulence is uniform, so that we can apply a Fourier transformation with respect to the coordinates and reduce (2) to a system of linear differential equations for the Fourier components $\delta\psi^\alpha_k(t)$

$$\frac{\partial}{\partial t}\delta\psi_k^\alpha = \sum_{\beta, k'} H^{\alpha\beta}_{kk'}\{\psi_{k''}^\gamma(t)\}\,\delta\psi_{k'}^\beta. \qquad (3)$$

The behavior of the small perturbations $\delta\psi^\alpha_k(t)$ in time can be simply described in general form in only two opposite limiting cases; 1) when the logarithmic decrement of the small perturbations is considerably greater than the characteristic frequencies of the turbulent motions; and 2), conversely, when the turbulent background oscillates more rapidly than the small perturbations are damped.

The second case is encountered in problems of so-called weak turbulence. Since the latter problem has been solved in its most general form (see, for example, [2]), we shall confine ourselves to a detailed analysis of the first case, when

$$\frac{1}{\omega_l|\delta\psi|}\frac{\partial|\delta\psi|}{\partial t} \gg 1. \qquad (4)$$

When (4) is satisfied, we can solve Eq. (3), assuming that the coefficients $H^{\alpha\beta}_{kk'}$ are constant in time. Then the characteristic numbers λ for solutions of $\delta\psi^\alpha_k$ in the form $\sim e^{\lambda t}$ are found from the characteristic equation

$$|H^{\alpha\beta}_{kk'} - \lambda\delta_{kk'}\delta^{\alpha\beta}| = 0. \qquad (5)$$

Writing this out in explicit form

$$\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \ldots + a_N = 0 \qquad (6)$$

we can apply the Routh-Hurwitz conditions [3] to it directly; these are necessary and sufficient for the real part of all the characteristic numbers to be negative in virtue of the assumption concerning the stability of the turbulent state.[2]

In the cases of practical interest, the coefficients of Eq. (6) are real, so that all its roots are conjugate-complex in pairs. Accordingly, all its coefficients must be positive

$$a_1 > 0,\ a_2 > 0,\ \ldots,\ a_N > 0. \qquad (7)$$

Stronger constraints on the coefficients are imposed by the Routh-Hurwitz condition, consisting in the positiveness of the sequence N of first principal minors of the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & 1 & 0 & 0 & \ldots \\ a_3 & a_2 & a_1 & 1 & \ldots \\ a_5 & a_4 & \cdot & \cdot & \ldots \end{vmatrix}. \qquad (8)$$

[1] For finite Reynolds numbers the number of degrees of freedom of turbulent motion n, represented here by motions with n different periods, is also finite.

[2] These conditions will be a criterion of asymptotic stability in the sense of (4), which does not take into account phenomena of the parametric resonance type.

Apart from inequalities (7), (8), the sufficient sign of the nondegeneracy of the matrix $\| H_{kk'}^{\alpha\beta} - \lambda \delta_{kk'} \delta^{\alpha\beta} \|$, proposed by Hadamard [3], is also useful; the matrix will be nonsingular if

$$|\lambda - H_{kk}^{\alpha\alpha}| > \sum_{k', \beta}{}' |H_{kk'}^{\alpha\beta}|.$$

Here the prime attached to the summation sign indicates that the diagonal term is omitted. On the other hand, it is known that in developed turbulence it is sufficient to increase the Reynolds number only slightly for an additional unstable solution to appear [1]. Therefore, there is always an eigenvalue $\lambda = \lambda_-$ with a very small negative real part Re $\lambda_- \to 0$. Then the above Hadamard condition is known to be violated, i.e.,

$$\sum_{k', \beta}{}' |H_{kk'}^{\alpha\beta}| > \sqrt{|\lambda_-|^2 + |H_{kk}^{\alpha\alpha}|^2} > |H_{kk}^{\alpha\alpha}|. \tag{9}$$

In the above analysis of the stability of the steady turbulent state of a fluid, it was assumed that the energy distribution with respect to the Fourier components of the turbulent motion was stationary. In fact, this assumption is not always justified. Thus, for example, in the case of Helmholtz instability it is more reasonable to assume a stationary distribution of turbulent energy in the form of individual line vortices and consider the stability of this stationary distribution [4]. Therefore, in each specific case it is necessary to choose the more convenient variables for describing the turbulence.

In conclusion, we note that, as a rule, the conditions of stability of the turbulent state give only constraints from below on the amplitudes of the pulsations and constraints on the phases of the amplitudes, since the rate at which energy is pumped from the unstable modes is determined precisely by the value of the latter (for certain assumptions concerning the phases of the amplitudes it is also possible to obtain constraints on the amplitudes from above). In this sense it would be useful to introduce thermodynamic considerations relating to the minimum production of entropy [5], which, probably, would give a constraint on the amplitudes from above. However, the introduction of the concept of entropy always requires the introduction of a certain rule of averaging in relation to the pattern of developed turbulence, which cannot always be done rationally.

As an illustration of the method described above, we shall consider the turbulent convection of an incompressible fluid in the gravitational field g. This is described by the equations of continuity, motion, and heat balance [6]:

$$\nabla \cdot \mathbf{v} = 0,$$

$$\frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \frac{p}{\rho} + g\alpha T'\mathbf{h} + \nu \nabla^2 \mathbf{v} \qquad \left(\alpha = \frac{1}{T}\right), \tag{10}$$

$$\frac{\partial T'}{\partial T} + \mathbf{v} \cdot \nabla T' = \varkappa \nabla^2 T' + \beta \mathbf{v} \cdot \mathbf{h} \qquad (\beta = -(\nabla T - \nabla_{ad} T) \cdot \mathbf{h}).$$

Here v, p, T, ρ are the hydrodynamic velocity vector, pressure, temperature, and the density of the medium, respectively, β is the superadiabatic gradient of the average temperature, T' is the temperature pulsation, ν, $\varkappa$ are the coefficients of viscosity and thermal conductivity. The pattern of developing turbulent motion will depend on two dimensionless parameters: the Prandtl number P and the Reynolds number R [7]:

$$P = \frac{\nu}{\varkappa} \ll 1, \qquad R = \frac{g\alpha\beta d^4}{\varkappa\nu (2\pi)^4} \gg 1. \tag{11}$$

Here d is the characteristic dimension of the convective layer.

Assuming that the Prandtl number P is small, we shall consider only the limiting case of very high thermal conductivity

$$RP \ll 1. \tag{12}$$

The nature of the constraints imposed by the conditions of stability of the turbulent background is quite well illustrated by the example of two-dimensional motion in the plane x, y (this is observed, for example, if a strong magnetic field H is applied along the z axis). In this case, system (10) is simplified and in the Fourier representation assumes the form

$$a_{kk} = -\left(\nu k^2 - \frac{g\alpha\beta [\mathbf{k} \times \mathbf{h}]^2}{\varkappa k^4}\right),$$

$$\delta \mathbf{v}_k = i [\mathbf{k} \times i_z] \delta\psi_k, \quad \frac{\partial}{\partial t} \delta\psi_k = \sum_{k'} a_{kk'} \delta\psi_{k'},$$

$$a_{kk'} = i (\mathbf{u}_{k-k'} \cdot \mathbf{k}) \left(1 - 2\frac{\mathbf{k}\mathbf{k}'}{k^2}\right), \qquad \mathbf{v} = \mathbf{u} + \delta\mathbf{v}. \tag{13}$$

From this we compute the coefficients of the characteristic equation (6)

$$a_1 = -\sum_{k} a_{kk} > 0, \qquad a_2 = \sum_{k \neq k'} [a_{kk} a_{k'k'} - a_{kk'} a_{k'k}] > 0, \tag{14}$$

$$a_3 = -\sum_{k \neq k' \neq k''} [a_{kk} a_{k'k'} a_{k''k''} - a_{kk} a_{k'k''} a_{k''k'} + a_{kk''} a_{k''k'} a_{k'k}] > 0. \tag{15}$$

The algorithm for computing the coefficients a_4, a_5 ... is clear from (14), (15). The summation in (14), (15) is carried out in the space of wave numbers

$$\text{from } |\mathbf{k}_0| = \frac{2\pi}{d} \quad \text{to} \quad |\mathbf{k}|_{\max} \gtrsim \frac{2\pi}{d} R^{1/4}.$$

We shall briefly consider the consequences of (14), (15) and conditions (8), (9). The law of conservation of energy in the steady-state turbulent regime

$$\sum_{k} a_{kk} |\mathbf{u}_k|^2 = 0 \tag{16}$$

in conjunction with (14) requires that the spectral energy density $|u_k|^2$ fall sharply in the direction of the shortwave end of the spectrum (especially in the region $a_{kk} > 0$). Condition (14) is identically fulfilled. The Hadamard criterion is an estimate from below for the amplitude of the longwave pulsations u_{k_2}. Assuming exponential behavior of the spectrum

$$\frac{k |\mathbf{u}_k|^2}{k_0 |\mathbf{u}_{k_0}|^2} = \left(\frac{k_0}{k}\right)^m \qquad \left(k_0 = \frac{2\pi}{d}\right) \tag{17}$$

this criterion may be written in the form

$$|\mathbf{u}_{k_0}| \gtrsim \frac{g\alpha\beta d^3}{\varkappa (2\pi)^3} \left[1 + R^{1/4(3-m)}\right]^{-1}. \tag{18}$$

Note that, from the point of view of applications, it is precisely the estimate of the pulsation energy $\mathbf{u}_{k_0}^2/2$ from below (18) that is important, since, from above, it is bounded by the work increment $g\alpha\Delta Td$ associated with movement of a macroscopic volume of dimension ~d through a distance ~d. Assuming that the uncompensated temperature drop $\Delta T \approx \beta u_{k_0} \tau_{k_0}$ ($\tau_{k_0} \sim 1/\varkappa k_0^2$), where τ_{k_0} is the dissipation time for the temperatue gradient due to head conduction, we get [7]

$$|\mathbf{u}_{k_0}| \lesssim \frac{g\alpha\beta d^3}{\varkappa (2\pi)^2}. \tag{19}$$

Further, from inequality (15) and the first of the Routh-Hurwitz conditions (8)

$$a_1 a_2 - a_3 > 0$$

it follows that minimal constraints are imposed on the phases of the three amplitudes $a_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}'\mathbf{k}''}a_{\mathbf{k}''\mathbf{k}}$, if the sum of these products in (15) is assumed negative. In this case (15) takes the form

$$\sum_{\mathbf{k}}\left(\nu k^2 - \frac{g\alpha\beta\,|\mathbf{k}\times\mathbf{h}|^2}{\varkappa k^4}\right) \gtrsim$$

$$\gtrsim \operatorname{Im}\sum_{\mathbf{q}_1\mathbf{q}_2}(\mathbf{u}_{\mathbf{q}_1}\cdot\mathbf{u}^*_{\mathbf{q}_1+\mathbf{q}_2})(\mathbf{u}_{\mathbf{q}_2}\cdot\mathbf{q}_1)\left(\sum_{\mathbf{q}}|\mathbf{u}_{\mathbf{q}}|^2\right)^{-1} > 0. \qquad (20)$$

Here, in accordance with (18), we neglect the first sum in (15) and consider that the main contribution to the sum with respect to the products of the three amplitudes is made by the region of wave numbers $k \gg q_1 = |k - k'|$, $q_2 = |k' - k''|$. This follows from the fact that the spectral energy density $|u|_q^2$ is practically equal to zero for $q \sim |\mathbf{k}|_{max} \sim (g\alpha\beta / \varkappa\nu)^{1/4}$.

Estimating the degree of randomness of the phases of the amplitudes with a certain coefficient

$$\operatorname{Im}\sum_{\mathbf{q}_1,\mathbf{q}_2}(\mathbf{u}_{\mathbf{q}_1}\cdot\mathbf{u}^*_{\mathbf{q}_1+\mathbf{q}_2})(\mathbf{u}_{\mathbf{q}_2}\cdot\mathbf{q}_1) = \varepsilon\sum_{\mathbf{q}_1,\mathbf{q}_2}|(\mathbf{u}_{\mathbf{q}_1}\cdot\mathbf{u}^*_{\mathbf{q}_1+\mathbf{q}_2})(\mathbf{u}_{\mathbf{q}_1}\cdot\mathbf{q}_1)|, \qquad (21)$$

we rewrite (20) in the form

$$\varepsilon \lesssim \frac{1}{|\mathbf{U}_{k_0}|}\,\frac{g\alpha\beta d^3}{7\,(2\pi)^2}\,\frac{[1 + R^{1/4(1-m)}]}{[1 + R^{1/8(7-3m)}]}\ln R^{1/4}. \qquad (22)$$

Note that the result does not contradict the work of E. Hopf [8], who showed that a normal law of velocity distribution is incompatible with the Kolmogorov spectrum and, consequently, $\varepsilon > R^{-1/2}$ for m = 5/3.

We obtain a numerical estimate for ε by substituting in (22) a certain spectral exponent m, using inequality (18) for an estimate of $|U_{k_0}|$. Inequality (22) can also be used in another respect. Thus, if we assume that the velocity distribution is strongly correlated, i.e., that ε ~1, then from (18), (22) we obtain a constraint on the amplitude from above. Finally, we note that a more detailed investigation of conditions (7), (8) would permit the strengthening of the inequalities (18), (22).

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5 September 1964 Novosibirsk